

AFFINE FRACTALS AS BOUNDARIES AND THEIR HARMONIC ANALYSIS

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ABSTRACT. We introduce the notion of boundary representation for fractal Fourier expansions, starting with a familiar notion of spectral pairs for affine fractal measures. Specializing to one dimension, we establish boundary representations for these fractals. We prove that as sets these fractals arise as boundaries of functions in closed subspaces of the Hardy space H^2 . By this we mean that there are lacunary subsets Γ of the non-negative integers, and associated closed Γ -subspace in the Hardy space $H^2(\mathbb{D})$, \mathbb{D} denoting the disk, such that for every function f in $H^2(\Gamma)$, and for every point z in \mathbb{D} , $f(z)$ admits a boundary integral represented by an associated measure μ , with integration over $\text{supp}(\mu)$ placed as a Cantor subset on the circle $\mathbb{T} := \text{bd}(\mathbb{D})$.

We study families of pairs: measures μ and sets Γ of lacunary form, admitting lacunary Fourier series in $L^2(\mu)$; i.e., configurations Γ arranged with a geometric progression of empty spacing, or missing parts, gaps. Given Γ , we find corresponding generalized Szegő kernels G_Γ , and we compare them to the classical Szegő kernel for \mathbb{D} .

Rather than the more traditional approach of starting with μ , and then asking for possibilities for sets Γ , such that we get Fourier series representations, we turn the problem upside down; now starting instead with a countably infinite discrete subset Γ , and, within a new duality framework, we study the possibilities for choices of measures μ .

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1. INTRODUCTION

In earlier papers, a number of authors studied a family of fractals X , and associated measures μ which arise as limits of iterated function systems (IFS). This framework includes for example infinite convolutions, and therefore Bernoulli measures.

The starting point is a finite family F of affine contractive mappings, and the measure μ then results as a consequence of a procedure of Hutchinson [Hut81]. The fractal X will be the support of μ . When the family F , is suitably restricted, it was shown in [JP98, Str00, Str98, LW02, DJ06]

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that the Hilbert space $L^2(\mu)$ then possesses a Fourier basis of orthogonal exponentials $\{e_\lambda : \lambda \in \Lambda\}$. The set Λ of exponentials in such an orthogonal basis will be called the *spectrum* of μ .

When a spectrum Λ exists we say that (μ, Λ) is a *spectral pair*, and there is a variety of results dealing with inverse spectral theory in this setting. Indeed, these results have many applications as they open up for the use of tools from Fourier analysis in the study of this family of fractals. While the procedure was developed for fractal measures μ with compact support in \mathbb{R}^d , for any d , there are a number of features that set aside the case $d = 1$, which will be the focus here. In this case, a normalization may be chosen in such a way that the spectrum Λ is contained in the non-negative integers \mathbb{N}_0 . So when (μ, Λ) is a spectral pair, and Λ is chosen in this way, we get a natural isometric embedding of $L^2(\mu)$ into the Hardy space $H^2(\mathbb{D})$ of analytic functions on the complex disk \mathbb{D} .

In this paper we deal with the resulting boundary representations. This study requires tools different from the classical theory. To see this note that the support X of μ may be placed on the boundary \mathbb{T} (one-torus) of the disk. But in the fractal cases, X has Lebesgue measure zero; recall the normalized Lebesgue measure is Haar measure of \mathbb{T} . By contrast, the classical boundary limits for functions in H^2 (Markov-Primalov-Fatou) yield only boundary limits almost everywhere (a.e) w.r.t. Lebesgue measure on \mathbb{T} . Indeed, our measures μ are typically singular with respect to Lebesgue measure, and have Lebesgue measure zero. Nonetheless we prove that the fractals arise as boundaries of closed subspaces $H^2(\Lambda)$ in the Hardy space H^2 . To do this we develop a family of reproducing kernels needed for the purpose. Our kernels have infinite product representations.

A separate motivation for our paper comes from the study of systems of frame vectors in Hilbert space. Frames generalize more familiar notions of bases in Fourier analysis; see for example [CF09, CW08]. Our focus here is on the case when both the Hilbert space and the choice of vectors are restricted. We take $L^2(\mu)$ for Hilbert space, and we take the vectors (functions) in $L^2(\mu)$ to be the familiar complex exponentials of Fourier analysis; hence Fourier frames. In some cases, we will arrive at orthogonal families, and in others not.

It was recently discovered that an important problem in operator algebras, the Kadison-Singer conjecture [KS59] is equivalent to intriguing open problems for frames, many with direct applications to signal processing; see e.g., [CW08]; and further section 4 below for further details.

Our present restricted context for frame computations appears to be a fertile ground for generating the kind of singular frames that are likely to have a bearing on Kadison-Singer in its frame incarnations. There are relatively more technical details involved in the search for examples of Fourier frames satisfying one or the other in the list of a priori frame estimates in the literature. While our main results regarding boundary representations are of independent interest, we hope that they will also serve to throw light on important questions regarding Fourier frames.

We will use the following definitions:

Definition 1.1. Let R be a $d \times d$ expansive real matrix, i.e., all its eigenvalues have absolute value strictly bigger than one. Let B be a finite subset of \mathbb{R}^d . We define the *affine iterated function system (IFS)* denoted (R, B) :

$$(1.1) \quad \tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B)$$

The unique Borel probability measure μ_B with the property that

$$(1.2) \quad \mu_B(E) = \frac{1}{\#B} \sum_{b \in B} \mu_B(\tau_b^{-1}(E)),$$

for all Borel sets in \mathbb{R}^d is called *the invariant measure* for the affine IFS (R, B) (see [Hut81]) for details.

Definition 1.2. Let R be a $d \times d$ matrix, and B, L two finite subsets of \mathbb{R}^d . We call (R, B, L) a *Hadamard system* if $\#B = \#L$ and the matrix

$$(1.3) \quad \frac{1}{\sqrt{\#B}} \left(e^{2\pi i R^{-1}b \cdot l} \right)_{b \in B, l \in L}$$

is unitary.

2. KERNELS FOR SUBSPACES OF THE HARDY SPACE

In this section we introduce the notion of boundary representation, and we prove that spectral pairs in one dimension admit such representations. By this we mean that when a spectral pair (μ, Γ) , in a general class, is given, then for every function f in the Γ -subspace in the Hardy space H^2 of the disk \mathbb{D} , and for every point z in \mathbb{D} , $f(z)$ admits a representation by a Γ -Szegő kernel G_Γ , with integration over $\text{supp}(\mu)$ placed as a Cantor subset on the circle $\mathbb{T} := \text{bd}(\mathbb{D})$. Thus $\text{supp}(\mu)$ placed on \mathbb{T} will be a boundary for the subspace $H^2(\Gamma)$, and integration is with respect to the fractal measure μ from the spectral pair.

We then turn to families of spectral pairs given by sets Γ of lacunary form, i.e., configurations arranged with a geometric progression of empty spacing, or a missing parts, gaps; lacunary Fourier series. For this case we show that our Szegő kernel G_Γ arises as a factor in the familiar and classical Szegő kernel for \mathbb{D} .

Definition 2.1. Following [Arv98] and [Rud87] we set $H^2 = H^2(\mathbb{D})$ the space of analytic functions in \mathbb{D}

$$f(z) = c_0 + c_1 z + c_2 z^2 + \dots, \quad (z \in \mathbb{D})$$

such that

$$\sum_{n \in \mathbb{N}_0} |c_n|^2 =: \|f\|_{H^2}^2 < \infty.$$

With the Szegő kernel

$$k(z, \xi) := \frac{1}{1 - \bar{z}\xi} \quad (\in H^2), z, \xi \in \mathbb{D};$$

we then get

$$(2.1) \quad f(z) = \langle k(z, \cdot), f \rangle_{H^2}$$

valid for all $f \in H^2$ and all $z \in \mathbb{D}$. The relation (2.1) is a simple instance of a reproducing kernel property. For the theory of reproducing kernels; see [Aro50], and also [Arv98, AL08, ADV09] for a variety of applications.

Theorem 2.2. Let μ be a probability measure on \mathbb{R} and assume $\Gamma \subset \mathbb{N}_0 := \{0, 1, 2, \dots\}$ is a spectrum for μ . Then

(i) The map $J : L^2(\mu) \rightarrow H^2$

$$(2.2) \quad J e_\gamma = z^\gamma, \quad (\gamma \in \Gamma)$$

extends to an isometric embedding of $L^2(\mu)$ into H^2 .

(ii) Define the map G on $\mathbb{D} \times \mathbb{R}$

$$(2.3) \quad G(z, x) := \sum_{\gamma \in \Gamma} \bar{z}^\gamma e_\gamma(x), \quad (z \in \mathbb{D}, x \in \mathbb{R})$$

Then

$$(2.4) \quad (Jf)(z) = \int f(x) \bar{G}(z, x) d\mu(x) = \langle G(z, \cdot), f \rangle_{L^2(\mu)}, \quad (z \in \mathbb{D})$$

(iii) Assume in addition that $\Gamma = R\Gamma + L$ for some $R \in \mathbb{N}, R \geq 2$ and some finite set $L \subset \mathbb{N}_0$ such that no two elements in L are congruent modulo R . Then

$$(2.5) \quad \bar{G}(z, x) = \prod_{n=0}^{\infty} \left(\sum_{l \in L} z^{R^n l} \bar{e}_l(R^n x) \right), \quad (z \in \mathbb{D}, x \in \mathbb{R}).$$

The infinite product is uniformly convergent for z in a compact subsets of \mathbb{D} and $x \in \mathbb{R}$.

Proof. Since $\{e_\gamma : \gamma \in \Gamma\}$ is an orthonormal basis in $L^2(\mu)$, (i) follows immediately.

(ii) Let

$$(2.6) \quad k_z(\xi) = \frac{1}{1 - \bar{z}\xi} \quad (z \in \mathbb{D}, \xi \in \mathbb{T})$$

be the Szegő kernel. We know that functions $F \in H^2$, can be recovered from their boundary values $F^\#$ by

$$(2.7) \quad F(z) = \int_{\mathbb{T}} \overline{k_z(\xi)} F^\#(\xi) d\xi = \langle k_z(\cdot), F \rangle_{H^2}, \quad (z \in \mathbb{D})$$

Let $G(z, \cdot) := (J^* k_z)(\cdot)$ for $z \in \mathbb{D}$. We have for $z \in \mathbb{D}$:

$$\langle G(z, \cdot), f \rangle_{L^2(\mu)} = \langle J^* k_z, f \rangle_{L^2(\mu)} = \langle k_z, Jf \rangle_{H^2} = (Jf)(z).$$

It remains to prove (2.3).

For $\gamma \in \Gamma$ and $z' \in \mathbb{D}$:

$$\langle e_\gamma, G(z', \cdot) \rangle_{L^2(\mu)} = \langle e_\gamma, J^* k_{z'} \rangle_{L^2(\mu)} = \langle J e_\gamma, k_{z'} \rangle_{H^2} = \langle z'^\gamma, k_{z'} \rangle_{H^2} = \overline{(z')^\gamma}.$$

Thus

$$G(z', \cdot) = \sum_{\gamma \in \Gamma} \overline{(z')^\gamma} e_\gamma(\cdot).$$

(iii) The condition implies that $0 \in \Gamma$; otherwise take $a = \min \Gamma$ and since $a = Ra' + l$, we must have $a' \leq a$, $a' \in \Gamma$, so $a = a' = 0$. Since the elements of L are incongruent modulo R , it follows that every $\gamma \in \Gamma$ can be written *uniquely* as $\gamma = R\gamma' + l$ for some $\gamma' \in \Gamma$ and $l \in L$. Then we have

$$\bar{G}(z, x) = \sum_{l \in L} \sum_{\gamma \in \Gamma} z^{R\gamma + l} \overline{e_{R\gamma + l}(x)} = \sum_{l \in L} z^l \overline{e_l(x)} \sum_{\gamma \in \Gamma} z^{R\gamma} \overline{e_\gamma(Rx)} = \left(\sum_{l \in L} z^l \overline{e_l(x)} \right) G(z^R, Rx).$$

Since

$$|G(z, x) - 1| \leq \sum_{\gamma \in \Gamma \setminus 0} |z|^\gamma \leq \sum_{n \geq 1} |z|^n = \frac{|z|}{1 - |z|}$$

for all $z \in \mathbb{D}$ and $x \in \mathbb{R}$ it follows that $G(z, x) - 1$ converges to 0 as $z \rightarrow 0$ (since $0 \in \Gamma$), uniformly in $x \in \mathbb{R}$.

Iterating the previous equality, and since $G(z^{R^n}, x)$ converges to 1 exponentially fast and uniformly for z in a compact subset of \mathbb{D} and for $x \in \mathbb{R}$, (iii) follows. \square

Definition 2.3. (i) Let X be a compact subset of $[0, 1]$. We shall also consider X as a subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ via the mapping $x \mapsto e_1(x) = e^{2\pi i x}$. We will further consider restrictions of functions f defined on all of \mathbb{C} via the identification $f(e_1(x)) = \tilde{f}(x)$ where \tilde{f} is then a \mathbb{Z} -periodic function on the line \mathbb{R} , and we view both \mathbb{R} and \mathbb{T} embedded in \mathbb{C} in the usual way. The notation \tilde{f} will be implicit in the discussion below.

(ii) Let X be as above, and let μ be a Borel probability measure supported on X . Consider subsets Γ of \mathbb{N}_0 . Set

$$\mathfrak{A}_\Gamma := \left\{ \sum_{\gamma \in \Gamma}^{(\text{finite})} c_\gamma z^\gamma : (c_\gamma)_{\gamma \in \Gamma} \text{ is a finite set of coefficients} \right\}.$$

(iii) We say that the pair (μ, Γ) has a *boundary representation* if there is a kernel function $k = k_{(\mu, \Gamma)}$ subject to the following conditions

- (a) $k : \mathbb{D} \times X \rightarrow \mathbb{C}$.
- (b) For all $z \in \mathbb{D}$, $k(z, \cdot) \in L^2(X, \mu)$; and
- (c) For all $f \in \mathfrak{A}_\Gamma$, $z \in \mathbb{D}$ we have

$$(2.8) \quad f(z) = \int_X \overline{k(z, x)} \tilde{f}(x) d\mu(x) = \left\langle k(z, \cdot), \tilde{f} \right\rangle_{L^2(\mu)}.$$

We denote by $H^2(\Gamma)$ the subspace of H^2 spanned by the functions z^γ with $\gamma \in \Gamma$.

Proposition 2.4. Let (μ, Γ) be a spectral pair, and assume that $\Gamma \subset \mathbb{N}_0$; then this pair has a boundary representation with kernel $k = k_{(\mu, \Gamma)}$ given by

$$(2.9) \quad \overline{k}(z, x) = \sum_{\gamma \in \Gamma} \overline{e_\gamma(x)} z^\gamma.$$

Moreover then

$$(2.10) \quad \langle k_z, k_w \rangle_{L^2(\mu)} = \sum_{\gamma \in \Gamma} (z \overline{w})^\gamma, \quad (z, w \in \mathbb{D}).$$

The representation (2.8) for functions in \mathfrak{A}_Γ extends to $f \in H^2(\Gamma)$; moreover then $\tilde{f} \in L^2(X, \mu)$, and

$$(2.11) \quad \|f\|_{H^2} = \|\tilde{f}\|_{L^2(\mu)}.$$

Proof. If $f \in \mathfrak{A}_\Gamma$, we set $f(z) = \sum_{\gamma \in \Gamma} c_\gamma z^\gamma$, and note that the corresponding periodic function \tilde{f} (as a restriction) satisfies

$$(2.12) \quad \tilde{f}(x) = \sum_{\gamma \in \Gamma} c_\gamma e_\gamma(x).$$

But then by restrictions $\tilde{f} \in L^2(X, \mu)$ and

$$(2.13) \quad c_\gamma = \int_X \overline{e_\gamma(x)} \tilde{f}(x) d\mu(x), \quad (\gamma \in \Gamma)$$

This is the $L^2(\mu)$ -Fourier expansion implied by the assumption that (μ, Γ) is a spectral pair. Since the sum in (2.12) is finite, substitution of (2.13) yields

$$(2.14) \quad f(z) = \sum_{\gamma} c_{\gamma} z^{\gamma} = \sum_{\gamma} \int_X \overline{e_{\gamma}(x)} \tilde{f}(x) d\mu(x) z^{\gamma} = \int_X \sum_{\gamma} \overline{e_{\gamma}(x)} z^{\gamma} \tilde{f}(x) d\mu(x) = \int_X \overline{k(z, x)} \tilde{f}(x) d\mu(x),$$

which is the kernel representation.

The formula (2.10) follows if we make use of the ONB property of $\{e_{\gamma} : \gamma \in \Gamma\}$.

The argument further shows that formula (2.9) is the unique kernel function. The remaining properties follow from an application *mutatis mutandis* of the details in the proof of Theorem 2.2 above.

For functions $f \in H^2(\Gamma)$ by definition we have the unique representation

$$(2.15) \quad f(z) = \sum_{\gamma \in \Gamma} c_{\gamma} z^{\gamma}, \quad (z \in \mathbb{D})$$

with

$$(2.16) \quad \|f\|_{H^2}^2 = \sum_{\gamma \in \Gamma} |c_{\gamma}|^2 < \infty.$$

But since (μ, Γ) is a spectral pair, we have (by Parseval applied to $L^2(\mu)$):

$$(2.17) \quad \sum_{\gamma \in \Gamma} |c_{\gamma}|^2 = \int_X |\tilde{f}|^2 d\mu$$

and

$$(2.18) \quad \tilde{f}(x) = \sum_{\gamma \in \Gamma} c_{\gamma} e_{\gamma}(x)$$

holds as an $L^2(\mu)$ -identity. This implies also (2.11).

But (2.14) also holds μ -a.e. when we pass to truncated summations on the right-hand side in (2.18).

It remains to justify the exchange of summation and integration for the computation (2.14) when f is now in $H^2(\Gamma) \subset H^2$.

Now let $f \in H^2(\Gamma)$, and let $(c_{\gamma})_{\gamma \in \Gamma}$ be the corresponding coefficients, see (2.13) and (2.15). Using again Parseval in the form (2.11), if $\epsilon > 0$, there is a finite subset $F \subset \Gamma$ such that

$$(2.19) \quad \sum_{\gamma \in \Gamma \setminus F} |c_{\gamma}|^2 < \epsilon.$$

Let $f_F = \sum_{\gamma \in F} c_{\gamma} z^{\gamma}$. Then for $|z| < 1$, using (2.8) for f_F :

$$\begin{aligned} & \left| \int \overline{k(z, x)} \tilde{f}(x) d\mu(x) - \sum_{\gamma \in F} c_{\gamma} z^{\gamma} \right| \leq \left| \int \overline{k(z, x)} \tilde{f}(x) d\mu(x) - \int \overline{k(z, x)} \tilde{f}_F(x) d\mu(x) \right| \\ & \quad + \left| \int \overline{k(z, x)} \tilde{f}_F(x) d\mu(x) - \sum_{\gamma \in F} c_{\gamma} z^{\gamma} \right| \\ & \leq \|k_z\|_{L^2(\mu)} \|\tilde{f} - \tilde{f}_F\|_{L^2(\mu)} + 0 = \|k_z\|_{L^2(\mu)} \|f - f_F\|_{H^2} \rightarrow 0 \text{ as } F \nearrow \Gamma. \end{aligned}$$

□

Example 2.5. There are differences between the boundary representation in the two cases, classical vs fractal, as we will see here with a simple example from [JP98]. Referring to the Cantor construction with scale 4, we get a spectral pair (μ, Γ) . We consider the monomial $f(z) = z^2$ not in the Γ -subspace subspace $H^2(\Gamma)$ in H^2 of the disk \mathbb{D} . We sketch how z^2 is represented by the Γ -Szegő kernel with integration over $\text{supp}(\mu)$ placed as a Cantor subset on the circle \mathbb{T} , and how it differs from the classical counterpart.

We caution that the representation of functions $f \in H^2(\Gamma)$ may differ from the more familiar H^2 -boundary corresponding to the Haar (normalized Lebesgue) measure on \mathbb{T} . The purpose of this example is to illustrate the significance of the isometric operator J in (2.2) in Theorem 2.2. Indeed the simple formula (2.2) is only valid for $\gamma \in \Gamma$. If $n \in \mathbb{N}_0 \setminus \Gamma$, then the function e_n will typically be the boundary for a function different than z^n .

To see this, take (μ, Γ) as follows: let μ be the invariant measure for the affine IFS with $R = 4$ and $B = \{-1, 1\}$, (see Definition 1.1 below) and let $\Gamma := \{\sum_{k=0}^n 4^k l_k : l_k \in \{0, 1\}, n \in \mathbb{N}_0\}$. Then (μ, Γ) is a spectral pair [JP98] and, for the Fourier transform, we have

$$(2.20) \quad \hat{\mu}(t) = \prod_{n=1}^{\infty} \cos\left(\frac{2\pi t}{4^n}\right), \quad (t \in \mathbb{R})$$

But then

$$J(e_2) = J\left(\sum_{\gamma \in \Gamma} \langle e_{\gamma}, e_2 \rangle e_{\gamma}\right) = \sum_{\gamma \in \Gamma} \hat{\mu}(2 - \gamma) z^{\gamma} \neq z^2.$$

A simple inspection of (2.20) shows that $\hat{\mu}(x) = \hat{\mu}(-x)$ for $x \in \mathbb{R}$, $\hat{\mu}(2) < 0$, $\hat{\mu}(2 - 16) \neq 0$ etc. Moreover, $\hat{\mu}$ vanishes on odd integers.

Definition 2.6. We say that Γ is a *Riesz sequence* if there are constants $0 < A_0 \leq A_1 < \infty$ such that: For all finite subsets of Γ and all finitely indexed subsets $\{c_{\gamma}\} \subset \mathbb{C}$, we have

$$(2.21) \quad A_0 \sum_{\gamma \in \Gamma} |c_{\gamma}|^2 \leq \sum_{\gamma, \gamma' \in \Gamma} \bar{c}_{\gamma} c_{\gamma'} \hat{\mu}(\gamma' - \gamma) \leq A_1 \sum_{\gamma \in \Gamma} |c_{\gamma}|^2.$$

See also Proposition 4.1 below and [DHSW10a].

Returning to the operator J from (2.2), but in a more general framework than Theorem 2.2, we have the following

Lemma 2.7. *Let μ be a probability measure supported on \mathbb{T} , and let Γ be a Riesz sequence, and set $Je_{\gamma} = z^{\gamma}$ for all $\gamma \in \Gamma$, see (2.2). Then J extends to a bounded operator $L^2(\mu) \rightarrow H^2$ if and only if the lower estimate in (2.21) holds for some $A_0 > 0$. In that case*

$$(2.22) \quad \|J\|_{L^2(\mu) \rightarrow H^2} \leq A_0^{-\frac{1}{2}}.$$

Moreover, there is a bounded inverse operator $J^{-1} : H^2(\Gamma) \rightarrow L^2(\mu)$, $J^{-1}(z_{\gamma}) = e_{\gamma}$, ($\gamma \in \Gamma$) if and only if the upper estimate in (2.21) holds for some $A_1 < \infty$. In that case

$$(2.23) \quad \|J^{-1}\|_{H^2(\Gamma) \rightarrow L^2(\mu)} \leq \sqrt{A_1}.$$

Proof. This is standard operator theory. □

Proposition 2.8. *Let μ and Γ be as specified above, and assume that the Riesz sequence estimate (2.21) holds. Let J and J^{-1} be the two bounded operators from Lemma 2.7. Then the measure space $(\text{supp}(\mu), \mathcal{B}, \mu)$ offers a boundary representation for $H^2(\Gamma) (\subset H^2)$, i.e., we get for all $f \in H^2(\Gamma)$, and all points $z \in \mathbb{D}$*

$$(2.24) \quad f(z) = \int_{\text{supp}(\mu)} \overline{(J^*k_z)}(x) (J^{-1}f)(x) d\mu(x)$$

where k_z in (2.24) denotes the Szegő kernel (2.6)

Proof. By virtue of the assumption on the pair (μ, Γ) , we get the two bounded operators J and J^{-1} as in Lemma 2.7. Below we then compute adjoint operators with respect to the two Hilbert inner products in H^2 , and in $L^2(\mu)$ respectively, denoted $\langle \cdot, \cdot \rangle_{H^2}$ and $\langle \cdot, \cdot \rangle_{L^2(\mu)}$ for emphasis.

Let $f \in H^2(\Gamma)$ and $z \in \mathbb{D}$ be fixed. Then

$$\begin{aligned} f(z) &= \langle k_z, f \rangle_{H^2} \text{ (by Szegő see Definition 2.1)} \\ &= \langle k_z, JJ^{-1}f \rangle_{H^2} \text{ (by Lemma 2.7)} \\ &= \langle J^*k_z, J^{-1}f \rangle_{L^2(\mu)} = \int_{\text{supp}(\mu)} \overline{(J^*k_z)}(x) (J^{-1}f)(x) d\mu(x), \end{aligned}$$

the desired conclusion (2.24). \square

Remark 2.9. The Γ -Szegő kernel in (2.24) is $G_\Gamma(z, \cdot) = J^*k_z$. Compare with the corresponding representation from Theorem 2.2; this is the special case of Proposition 2.8 for the case when (μ, Γ) is assumed to be a spectral pair, as opposed to merely a Riesz system.

In the theorem below, we consider spectral pairs given by sets Γ of lacunary form, i.e., configurations arranged with a choice of geometric progressions of empty spacing or gaps, similar to lacunary Fourier series. We then prove that our Szegő kernels G_Γ arise as factors in the familiar and classical Szegő kernel for \mathbb{D} .

We use the notation $A \oplus A' = \{0, \dots, R-1\}$ to indicate that every element $k \in \{0, \dots, R-1\}$ can be written uniquely as $k = a + a'$ with $a \in A$ and $a' \in A'$. We will also need the following Lemma:

Lemma 2.10. [Lon67] *Suppose $A \oplus A' = \{0, \dots, R-1\}$ then one the following affirmations is true*

- (i) $A = \{0\}$ or $A' = \{0\}$.
- (ii) $1 \in A$ and there exist a number $d \geq 2$ that divides R and two subsets C, C' of \mathbb{N}_0 such that $A = dC \oplus \{0, \dots, d-1\}$, $A' = dC'$ and $C \oplus C' = \{0, \dots, R/d-1\}$.
- (iii) $1 \in A'$ and (ii) holds with the roles of A and A' reversed.

Theorem 2.11. *Suppose A is a subset of \mathbb{N}_0 such that there exists $A' \subset \mathbb{N}_0$ and $R \in \mathbb{N}, R \geq 2$ such that $A \oplus A' = \{0, \dots, R-1\}$ and $A, A' \neq \{0\}$. Then*

- (i) *There exists finite subsets $L, L' \subset \{0, \dots, R-1\}$ such that $L \oplus L' = \{0, \dots, R-1\}$ and with the property that (R, A, L) and (R, A', L') are Hadamard systems. Also $\gcd(A)$ divides R . The set L can be picked such that $\gcd(A) \cdot \max(L) < R$. Similarly for L' . Here $\gcd(A)$ represents the greatest common divisor of A .*
- (ii) *Let μ_A be the invariant measure associated to the IFS (R, A) and similarly for $\mu_{A'}$. Then the convolution $\mu_A * \mu_{A'} = \lambda|_{[0,1]}$ = the Lebesgue measure restricted to $[0, 1]$.*

- (iii) μ_A is spectral with spectrum $\Gamma(L) = \{\sum_{k=0}^n R^k l_k : l_k \in L, n \in \mathbb{N}_0\}$ and similarly $\mu_{A'}$ is spectral with spectrum $\Gamma(L')$.
- (iv) The kernels satisfy the following relation

$$(2.25) \quad G_{\Gamma(L)} G_{\Gamma(L')} = k,$$

where k is the classical Szegő kernel.

Proof. (i) We proceed by induction on R . For $R = 2$ this is trivial.

We use Lemma 2.10. The case (i) in this Lemma cannot occur under our hypotheses. Assume we are in the case (ii) of Lemma 2.10, case (iii) can be treated similarly. According to the induction hypothesis, there exist sets M, M' such that $M \oplus M' = \{0, \dots, R/d - 1\}$ and $(R/d, C, M)$ and $(R/d, C', M')$ are Hadamard systems.

Define $L := M \oplus \frac{R}{d}\{0, \dots, d-1\}$ and $L' = M'$. Then it is easy to see that $L \oplus L' = \{0, \dots, R-1\}$ and (R, A', L') is a Hadamard system (since $A' = dC'$). It remains to check that (R, A, L) is a Hadamard system. We use the fact that $A = dC \oplus \{0, \dots, d-1\}$. Take $m, m' \in L$ and $j, j' \in \{0, \dots, d-1\}$. Then

$$\sum_{c \in C} \sum_{i=0}^{d-1} e^{i2\pi \frac{1}{R}(dc+i)((m-m')+\frac{R}{d}(j-j'))} = \sum_{i=0}^{d-1} e^{i2\pi \frac{i}{R}((m-m')+\frac{R}{d}(j-j'))} \sum_{c \in C} e^{i2\pi \frac{1}{R/d}c(m-m')},$$

because $\frac{1}{R}dc\frac{R}{d}(j-j')$ is an integer.

If $m \neq m'$, then using the fact the $(R/d, C, M)$ is a Hadamard system, we get that the sum is zero. If $m = m'$ and $j \neq j'$ then using the fact that $(d, \{0, \dots, d-1\}, \{0, \dots, d-1\})$ is a Hadamard system, we get again that the sum is zero.

This proves that (R, A, L) is a Hadamard system.

It remains to prove the last statement. We proceed also by induction on R . If $R = 4$ the result is easy to obtain. Since $A \neq \{0\}$ we have either $1 \in A$ (and this case is trivial) or $A = dC$ for some $d \geq 2$, and $C \oplus C' = \{0, \dots, R/d - 1\}$. Then as before, for the pair we can pick the dual sets (M, M') , and we can pick $L = M$. By the induction hypothesis $\gcd(C) \cdot \max(M) < R/d$ so $\gcd(A) \cdot \max(L) = d \gcd(C) \cdot \max(M) < R$. Also by the induction hypothesis $\gcd(C)$ divides R/d so $\gcd(A) = d \gcd(C)$ divides R .

(ii) Let

$$\chi_A(x) = \frac{1}{\#A} \sum_{a \in A} e^{2\pi i a x}, \quad \chi_{A'}(x) = \frac{1}{\#A'} \sum_{a' \in A'} e^{2\pi i a' x}$$

Since $A \oplus A' = \{0, \dots, R-1\}$ we can see that

$$\chi_A(x) \chi_{A'}(x) = \frac{1}{R} \sum_{j=0}^{R-1} e^{2\pi i j x} =: \chi_{\{0, \dots, R-1\}}(x).$$

Then, using the infinite product formula for $\widehat{\mu}_A$ (see [DJ06]),

$$\begin{aligned} \widehat{\mu_A * \mu_{A'}}(x) &= \widehat{\mu}_A(x) \widehat{\mu}_{A'}(x) = \prod_{n=1}^{\infty} \chi_A(R^{-n}x) \prod_{n=1}^{\infty} \chi_{A'}(R^{-n}x) = \prod_{n=1}^{\infty} \chi_A(R^{-n}x) \chi_{A'}(R^{-n}x) \\ &= \prod_{n=1}^{\infty} \chi_{\{0, \dots, R-1\}}(R^{-n}x). \end{aligned}$$

The change in the order of multiplication is allowed since the infinite products are uniformly convergent on compact sets.

The Lebesgue measure on $[0, 1]$ is the invariant measure associated to the affine IFS $(R, \{0, \dots, R-1\})$. Therefore the last product above is the Fourier transform of $\lambda|_{[0,1]}$.

(iii) Using the results from [DJ06, Theorem 8.4], we have to show there are no non-trivial extreme cycles (or χ_A -cycles as they are called in [DJ06]). Recall that a non-trivial extreme cycle is a finite set of non-zero points $\{x_0, x_1, \dots, x_{p-1}, x_p := x_0\}$ such that there exist $l_i \in L$ with $R^{-1}(x_i + l_i) = x_{i+1}$ for all $i \in \{0, \dots, p-1\}$, and such that $|\chi_A(x_i)| = 1$ for all $i \in \{0, \dots, p-1\}$.

Assume by contradiction that there is such an extreme cycle. Since $|\chi_A(x_i)| = 1$, and since $0 \in A$, we must have equality in the triangle inequality so $e^{2\pi i a x_i} = 1$, which means that $x_i \in \frac{1}{g}\mathbb{Z}$, with $g = \gcd(A)$. Consider the smallest non-zero cycle, say $x_0 = k/g$. Then $\frac{1}{R}(\frac{k}{g} + l)$ is also a cycle point for some $l \in L$, so it is also of the form k'/g . From (i) we know $gl < R$.

First, if $k \geq 2$ then

$$\frac{k'}{g} = \frac{1}{R}(\frac{k}{g} + l) < \frac{1}{R}(\frac{k}{g} + \frac{R}{g}) \leq \frac{Rk}{Rg} = \frac{k}{g} = x_0,$$

and this would contradict the fact that x_0 is the smallest non-zero cycle. Then $k = 1$, and using the computation above, with $k = 1$, we get $\frac{k'}{g} < \frac{2}{g}$, and therefore $k' = 1$ too. But then we must have

$$\frac{1}{R}(\frac{1}{g} + l) = \frac{1}{g}$$

so $1 + gl = R$. Since g divides R (see (i)), we obtain that $g = 1$. Then, since $A' \neq \{0\}$, we must have that $L' \neq \{0\}$ so $\max(L) < R - 1$ and we get a contradiction.

In conclusion there are no non-trivial extreme cycles, hence with [DJ06], we get that $\Gamma(L)$ is a spectrum for μ_A . Similarly for $\mu_{A'}$.

(iv) We have

$$G_{\Gamma(L)}(z, x)G_{\Gamma(L')}(z, x) = \sum_{\gamma \in \Gamma(L), \gamma' \in \Gamma(L')} \bar{z}^{\gamma+\gamma'} e_{\gamma+\gamma'}(x)$$

Note that $\Gamma(L) \oplus \Gamma(L') = \mathbb{N}_0$. This can be seen from the base R expansion of any natural number $n = \sum_{k=0}^p R^k a_k$, with $a_k \in \{0, \dots, R-1\}$. Since $L \oplus L' = \{0, \dots, R-1\}$, we get the unique decomposition as a sum $\gamma + \gamma'$ with $\gamma \in \Gamma(L)$ and $\gamma' \in \Gamma(L')$. Thus

$$G_{\Gamma(L)}(z, x)G_{\Gamma(L')}(z, x) = \sum_{n=0}^{\infty} \bar{z}^n e_n(x) = \frac{1}{1 - \bar{z}e_1(x)} = k(z, x).$$

□

3. SET-MEASURE DUALITY

Most earlier studies of classes of spectral pairs (μ, Γ) have started with μ , and then asked what possibilities there are for sets Γ that make the two into a spectral pair; i.e., allow a Fourier series representation, typically with lacunary Fourier frequencies. In much of this work, the measures μ have been chosen at the outset to be self-similarity defined by a finite family of affine maps. In this section, we turn the problem upside down; starting with a countably discrete subset Γ , we ask what the possibilities are for choices of μ . To do this we introduce a new duality framework.

Definition 3.1. Let the setting be as above, including dimension $d \geq 1$; and consider

$$(3.1) \quad \mathcal{M}_1 := \left\{ \mu : \mu \text{ is a Borel probability measure with compact support in } \mathbb{R}^d \right\}.$$

We equip \mathcal{M}_1 with its weak* topology; and consider $\Gamma \subset \mathbb{R}^d$ some countable discrete subset.

$$(3.2) \quad \mathcal{M}^\perp(\Gamma) := \left\{ \mu \in \mathcal{M}_1 : \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 \leq 1 \right\}.$$

If $A \geq 1$, set

$$(3.3) \quad \mathcal{M}_A(\Gamma) := \left\{ \mu \in \mathcal{M}_1 : \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 \leq A, \text{ for all } t \in \mathbb{R}^d \right\}.$$

$$(3.4) \quad \mathcal{M}^{OB}(\Gamma) := \left\{ \mu \in \mathcal{M}_1 : \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 = 1 \text{ for all } t \in \mathbb{R}^d \right\}.$$

Note that $\mathcal{M}^\perp(\Gamma) = \mathcal{M}_1(\Gamma)$.

Lemma 3.2. Fix Γ as in the definition. Then

- (i) $\mu \in \mathcal{M}^\perp(\Gamma)$ iff $\{e_\gamma\}_{\gamma \in \Gamma}$ is an orthogonal family in $L^2(\mu)$.
- (ii) $\mu \in \mathcal{M}^{OB}(\Gamma)$ iff $\{e_\gamma\}_{\gamma \in \Gamma}$ is an ONB in $L^2(\mu)$.
- (iii) If $\{e_\gamma\}_{\gamma \in \Gamma}$ forms a Bessel sequence in $L^2(\mu)$ with bound A , i.e.,

$$\sum_{\gamma \in \Gamma} \left| \langle e_\gamma, f \rangle_{L^2(\mu)} \right|^2 \leq A \|f\|_{L^2(\mu)}^2, \quad (f \in L^2(\mu)),$$

then $\mu \in \mathcal{M}_A(\Gamma)$.

Proof. For (i), (ii) see [DJ06]. (iii) follows by applying the Bessel estimate to the functions e_t . \square

Previously, the measures μ have been chosen at the outset to have self-similarity defined by a finite family of affine maps. In the theorem below, we turn the problem upside down, thus allowing for the possibility of any measure μ . Hence our starting point is a fixed countably discrete subset Γ , and we ask what the possibilities are for choices of μ .

Theorem 3.3. Fix Γ as in the definition, and $A \geq 1$. Then $\mathcal{M}_A(\Gamma)$ is a convex, weak*-compact subset of \mathcal{M}_1 . Same is true for $\mathcal{M}^\perp(\Gamma)$. The set $\mathcal{M}^{OB}(\Gamma)$ is contained in the extreme points of $\mathcal{M}^\perp(\Gamma)$.

Proof. Let $\mu_1, \mu_2 \in \mathcal{M}_A(\Gamma)$ and $\alpha \in [0, 1]$ and set $\mu_\alpha := \alpha\mu_1 + (1 - \alpha)\mu_2$; then using Schwarz's inequality on $l^2(\Gamma)$

$$\begin{aligned} \sum_{\gamma} |\widehat{\mu}_\alpha(t - \gamma)|^2 &\leq \alpha^2 A + (1 - \alpha)^2 A + 2\alpha(1 - \alpha) \operatorname{Re} \sum_{\gamma} \widehat{\mu}_1(t - \alpha) \widehat{\mu}_2(t - \alpha) \\ &\leq A(\alpha^2 + (1 - \alpha)^2) + 2\alpha(1 - \alpha) \left(\sum_{\gamma} |\widehat{\mu}_1(t - \gamma)|^2 \right)^{1/2} \left(\sum_{\gamma} |\widehat{\mu}_2(t - \gamma)|^2 \right)^{1/2} \\ &\leq A(\alpha^2 + (1 - \alpha)^2 + 2\alpha(1 - \alpha)) = A. \end{aligned}$$

Hence μ_α is in $\mathcal{M}_A(\Gamma)$ and $\mathcal{M}_A(\Gamma)$ is convex.

We check that $\mathcal{M}_A(\Gamma)$ is weak*-closed. Take $\mu_n \in \mathcal{M}_A(\Gamma)$, and $\mu_n \rightarrow \mu \in \mathcal{M}_1$. Then, since $\hat{\mu}(t) = \int e_t d\mu$, we have that $\lim_n \hat{\mu}_n(t) = \hat{\mu}(t)$ for all $t \in \mathbb{R}^d$. Using Fatou's lemma, we have

$$\sum_{\gamma} |\hat{\mu}(t - \gamma)|^2 \leq \liminf_n \sum_{\gamma} |\hat{\mu}_n(t - \gamma)|^2 \leq A,$$

so μ is in $\mathcal{M}_A(\Gamma)$, and therefore $\mathcal{M}_A(\Gamma)$ is weak*-closed hence compact.

Since $\mathcal{M}^\perp(\Gamma) = \mathcal{M}_1(\Gamma)$, the same holds for $\mathcal{M}^\perp(\Gamma)$.

We check that points in $\mathcal{M}^{OB}(\Gamma)$ are extreme in $\mathcal{M}^\perp(\Gamma)$. For this, consider μ_α, μ_1, μ_2 and α as in the beginning of the proof, $A = 1$, and assume $\mu_\alpha \in \mathcal{M}^{OB}(\Gamma)$; i.e.,

$$\sum_{\gamma} |\hat{\mu}_\alpha(t - \gamma)|^2 = 1, \quad (t \in \mathbb{R}^d)$$

see Lemma 3.2. Using the same calculation it follows that we have equalities in all inequalities. In particular (assuming $0 < \alpha < 1$), we have that

$$(3.5) \quad \sum_{\gamma} |\hat{\mu}_1(t - \gamma)|^2 = \sum_{\gamma} |\hat{\mu}_2(t - \gamma)|^2 = 1, \quad (t \in \mathbb{R}^d).$$

Also, we must have equality in the Schwarz inequality, so the vectors $(\hat{\mu}_1(t - \gamma))_{\gamma \in \Gamma}$ and $(\hat{\mu}_2(t - \gamma))_{\gamma \in \Gamma}$ in $l^2(\Gamma)$ are proportional, and since they both have norm one, the proportionality constant is $e^{i\theta}$ for some $\theta \in \mathbb{R}$. But the real part of the product must be equal to the absolute value, so $e^{i\theta} = 1$. This implies that $\hat{\mu}_1 = \hat{\mu}_2$ so $\mu_1 = \mu_2 = \mu_\alpha$. Hence μ is an extreme point. \square

In earlier papers dealing with spectral pairs in one dimension, for example [JP98] and [DJ06], one typically begins with a positive integer (> 1) defining a scale similarity, for example an infinite convolution as in Example 2.5 above. It is interesting to compare the two scale numbers 3 and 4 (the case in Example 2.5). If μ_3 is the Cantor measure (i.e., for the ternary case), then it was shown in [JP98] that $L^2(\mu_3)$ cannot have more than two orthogonal Fourier frequencies. By contrast, it was further shown in [JP98], that all the Cantor measures μ_m , for m even, are in the opposite extreme: they allow for spectra, i.e., admit sets Γ_m such that (μ_m, Γ_m) is a spectral pair. In the example below, we turn around the question: we begin with a ternary choice for the set Γ and then ask what possibilities there might be for μ .

Example 3.4. Set $d = 1$ and

$$\Gamma := \left\{ \sum_{i=0}^n a_i 3^i : a_i \in \{0, 1\}, n \in \mathbb{N}_0 \right\}.$$

Then $\mathcal{M}^\perp(\Gamma) = \mathcal{M}^\perp(\mathbb{Z})$ and $\mathcal{M}^{OB}(\Gamma) = \emptyset$.

To see this, take $\mu \in \mathcal{M}^\perp(\Gamma)$. Using the base-3 decomposition of positive integers using the digits $\{0, 1, -1\}$, we see that $\Gamma - \Gamma = \mathbb{Z}$. So $\hat{\mu}$ must vanish on $\mathbb{Z} \setminus \{0\}$. Therefore $\mu \in \mathcal{M}^\perp(\mathbb{Z})$. Since $2 \notin \Gamma$ and $e_2 \perp e_\gamma$ for all $\gamma \in \Gamma$, it follows that the set $\{e_\gamma : \gamma \in \Gamma\}$ cannot be complete.

4. CONCLUSIONS AND OPEN PROBLEMS

While the literature on frame systems in Hilbert space is vast, see for example [CF09, CW08], and even in Banach space [CC08], our focus here is on the case when both the Hilbert space and the choice of vectors are restricted. We take the Hilbert space to be $L^2(\mu)$ where the family of measures is as outlined above, and we take the vectors to be the complex exponentials of Fourier analysis; Fourier frames.

It was recently discovered that an important problem in operator algebras, the Kadison-Singer conjecture [KS59] is equivalent to important open problems for frames; see e.g., [CW08]. Our present restricted context for frame computations appears to be a fertile ground for generating the kind of singular frames that are likely to have a bearing on Kadison-Singer in its frame incarnations. But this means that there are relatively more technical details involved in the search for examples of Fourier frames satisfying one or the other of the frame estimates that subdivide the subject. Below we include a table of cases, and an overview of what is known, and what is still open.

Let μ be in \mathcal{M}_1 and Γ a discrete subset of \mathbb{R}^d . Define the function

$$(4.1) \quad \sigma_\Gamma(t) := \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2, \quad (t \in \mathbb{R}^d).$$

and let

$$(4.2) \quad E(\Gamma) := \{e_\gamma : \gamma \in \Gamma\}.$$

The function σ_Γ plays an central role in the study of sequences of exponential functions in \mathbb{R}^d . We review some of its properties here, and we list some open questions related to it.

Proposition 4.1. *Let μ and Γ be as above. The function σ_Γ has the following properties:*

- (i) $E(\Gamma)$ is an ONB for $L^2(\mu)$ if and only if $\sigma_\Gamma \equiv 1$.
- (ii) $E(\Gamma)$ is an orthonormal set in $L^2(\mu)$ if and only if $\sigma_\Gamma \leq 1$.
- (iii) $E(\Gamma)$ is a maximal orthonormal set of exponentials if and only if $0 < \sigma_\Gamma \leq 1$.
- (iv) If $E(\Gamma)$ is a Bessel sequence with bound $B > 0$ then $\sigma_\Gamma \leq B$.
- (v) If $E(\Gamma)$ is a frame with bounds $A, B > 0$ then $A \leq \sigma_\Gamma \leq B$.
- (vi) $E(\Gamma)$ is a Riesz basic sequence with bounds $A, B > 0$ if and only if the self-adjoint matrix

$$(4.3) \quad \mathcal{G}_\Gamma := (\widehat{\mu}(\gamma - \gamma'))_{\gamma, \gamma' \in \Gamma}$$

satisfies $AI_{l^2(\Gamma)} \leq \mathcal{G}_\Gamma \leq BI_{l^2(\Gamma)}$.

The statements (i), (ii), and (iv) just repeat Lemma 3.2. For a proof of (v), (vi) see [DJ06, DHSW10b, DHSW10a]. To prove (iii), we see that if $\sigma_\Gamma > 0$, and if $\gamma' \notin \Gamma$, then there is a $\gamma \in \Gamma$ such that $\widehat{\mu}(\gamma - \gamma') \neq 0$, so $e_{\gamma'}$ is not orthogonal to e_γ . Hence $E(\Gamma)$ is maximal. Conversely, if this set is maximal, then we cannot have $\sigma_\Gamma(t) = 0$ because that would imply that e_t is orthogonal to all e_γ with $\gamma \in \Gamma$.

Here are some known results related to Proposition 4.1. We denote by μ_4 the measure in the Jorgensen-Pedersen example [JP98], i.e., the invariant measure for the affine IFS with $R = 4$ and $B = \{0, 2\}$, and by μ_3 the middle third Cantor measure, i.e., $R = 3$, $B = \{0, 2\}$.

- (i) [JP98, DJ06, DHS09] There are infinitely many sets Γ that contain 0 such that $E(\Gamma)$ is an ONB for $L^2(\mu_4)$.
- (ii) [JP98] There are no sets Γ with 3 or more elements such that $E(\Gamma)$ is orthogonal in $L^2(\mu_3)$.

- (iii) [DHS09] There are maximal sets of orthogonal exponentials in $L^2(\mu_4)$ which are not ONBs for $L^2(\mu_4)$.
- (iv) [DHSW10a] There are sets Γ of positive Beurling dimension such that $E(\Gamma)$ is a Bessel sequence in $L^2(\mu_3)$.
- (v) [DHSW10a] There are sets Γ of positive Beurling dimension such that $E(\Gamma)$ is a Riesz basic sequence in $L^2(\mu_3)$.

Here is a list of questions belonging to the same circle of ideas:

Questions. The following question are still open at the time this paper was written:

- (i) Does the converse of Proposition 4.1 (iv) hold?
- (ii) Does the converse of Proposition 4.1 (v) hold?
- (iii) Are there any sets Γ such that $E(\Gamma)$ is a frame for $L^2(\mu_3)$?
- (iv) Are there any sets Γ such that $E(\Gamma)$ is a Riesz basis for $L^2(\mu_3)$?

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